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Bohr's phenomenon for analytic functions and the hyperbolic metric

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A link is established between Bohr's inequality for classes of analytic functions and the hyperbolic metric. The classes considered consist of analytic functions mapping the unit disk respectively into the right half-plane, the slit region, and to the exterior of the unit disk.

1 Introduction

Bohr's inequality states that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is analytic in the unit disk U and $|f(z)| < 1$ for all $z \in U$, then

$$\sum_{n=0}^{\infty} |a_n z^n| \leq 1 \tag{1.1}$$

for all $z \in U$ with $|z| \leq 1/3$. This inequality was discovered by Bohr [11] in 1914. Bohr actually obtained the inequality for $|z| \leq 1/6$. Wiener, Riesz and Schur, independently established the inequality for $|z| \leq 1/3$ and showed that the bound $1/3$ is sharp [15], [20], [21]. Other proofs were also given in [16]–[18]. Boas and Khavinson [10], and more recently Aizenberg [4]–[6] extended the inequality to several complex variables.

Bohr's inequality drew the attention of operator algebraists after Dixon [12] showed a connection between the inequality and the characterization of Banach algebras that satisfy Von Neumann's inequality. Specifically, by using Bohr's inequality, Dixon constructed an example of a Banach algebra that satisfies Von Neumann's inequality but is not isomorphic to the algebra of bounded operators on a Hilbert space. Paulsen and Singh [16] extended Bohr's inequality to Banach algebras.

A class of analytic (or harmonic) functions in the unit disk U is said to satisfy Bohr's phenomenon if an inequality of type (1.1) holds uniformly in $|z| < \rho_0$, for some $0 < \rho_0 \leq 1$, and for all functions in the class.

This article considers the class of functions subordinated to a given analytic function. For two analytic functions f and g in the unit disk U, the function g is subordinate to f if there exists a Schwarz function φ, analytic in U with $\varphi(0) = 0$ and $|\varphi(z)| < 1$, satisfying $g = f \circ \varphi$. In particular, when f is univalent, then g is subordinate to f provided $g(U) \subset f(U)$ and $g(0) = f(0)$ ([13, p. 190], [19, p. 35]). Consequently, when g is subordinate to f, then $|g'(0)| \leq |f'(0)|$. For additional details on subordination classes, see for example [13] or [19].

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Let $S(f)$ denote the class of functions g subordinate to a fixed function f and $f(U) = \Omega$. The class $S(f)$ is said to satisfy a Bohr's phenomenon if for any $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, there is a ρ_0, $0 < \rho_0 \leq 1$, so that

$$\sum_{n=1}^{\infty} |b_n z^n| \leq d(f(0), \partial\Omega) \tag{1.2}$$

for $|z| < \rho_0$. Here $d(f(0), \partial\Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of a domain Ω. Obviously, when $\Omega = U$, $d(f(0), \partial\Omega) = 1 - |f(0)|$ and in this case (1.2) reduces to (1.1).

It is known that $S(f)$ has a Bohr's phenomenon when f is univalent. Abu-Muhanna [1] recently showed that every $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$ satisfies

$$\sum_{n=1}^{\infty} |b_n z^n| \leq d(f(0), \partial\Omega) \tag{1.3}$$

for $|z| \leq \rho_0 = 3 - 2\sqrt{2} \cong 0.171\,57$. The radius ρ_0 is sharp for the Koebe function $f(z) = z/(1-z)^2$. In particular, when f is convex, it was shown in [6] that (1.3) remains valid for $\rho_0 = 1/3$, a result which includes (1.1) when $\Omega = U$. In a recent paper [2], we had investigated Bohr's inequality for functions mapping the unit disk into the exterior of a compact convex set.

This article studies Bohr's phenomenon to three classes of analytic functions mapping the unit disk respectively into the right half-plane, the slit region, and to the exterior of the unit disk. It is shown that Bohr's phenomenon is carried over to the hyperbolic metric. In other words, the hyperbolic metric can be used to describe the phenomenon. This is not surprising if one takes into account the invariant nature of the metric. Results of Section 2 are proved by using the properties of the hyperbolic metric and sharpened by introducing the hyperbolic metric into the conclusions.

Let us recall [8], [9] that the hyperbolic metric for U is defined by

$$\lambda_U(z)|dz| = \frac{2|dz|}{1 - |z|^2}, \tag{1.4}$$

the hyperbolic length by

$$L_U(\gamma) = \int_\gamma \lambda_U(z)|dz|,$$

and the hyperbolic distance by

$$d_U(z, w) = \inf_\gamma L_U(\gamma),$$

over all smooth curves γ joining z to w in U. The following four results from [8], [9] will be required.

Theorem 1.1 *The hyperbolic distance in U is given by*

$$d_U(z, w) = \log \frac{1 + \left|\frac{z-w}{1-z\overline{w}}\right|}{1 - \left|\frac{z-w}{1-z\overline{w}}\right|}.$$

Hence $d_U(z, 0) = \log \frac{1+|z|}{1-|z|} \to \infty$, as $|z| \to 1$. This shows that the disk together with the hyperbolic metric (the Poincare space) is a hyperbolic plane.

Theorem 1.2 (Schwarz-Pick) *Let $f : U \to U$ be analytic. Then*

$$d_U(f(z), f(w)) \leq d_U(z, w).$$

Equality is possible only when $f(z) = e^{i\theta} \frac{z+a}{1+\overline{a}z}$, $a \in U$, $\theta \in \mathbb{R}$.

For any simply connected domain Ω, and $f : U \to \Omega$ the corresponding conformal map, define the hyperbolic metric of Ω by

$$\lambda_\Omega(w) = \frac{\lambda_U(f^{-1}(w))}{|f'(f^{-1}(w))|}. \tag{1.5}$$

The metric λ_Ω is independent of the choice of the conformal map f used.

Theorem 1.3 *Let $g : U \to \Omega$ be analytic. Then*

$$d_\Omega(g(z), g(w)) \le d_U(z, w).$$

Equality is possible only when g is conformal onto Ω. In particular, $d_\Omega(g(z), g(0)) \le \log \frac{1+|z|}{1-|z|}$.

The following estimates on the hyperbolic metric λ_Ω in terms of the Euclidean distance will be of interest.

Theorem 1.4 *Let Ω be a simply connected proper domain, and $f : U \to \Omega$ be conformal. Then*

$$\frac{1}{2d(f(z), \partial\Omega)} \le \lambda_\Omega(f(z)) \le \frac{2}{d(f(z), \partial\Omega)}.$$

Equality holds on the left if and only if Ω is a slit-plane. If Ω is convex, then

$$\frac{1}{d(f(z), \partial\Omega)} \le \lambda_\Omega(f(z)) \le \frac{2}{d(f(z), \partial\Omega)}.$$

Equality holds on the left if and only if Ω is a half-plane.

2 Main results

This section contains the main results on Bohr's phenomenon which have been improved by incorporating the hyperbolic metric.

2.1 Half-planes and convex domains

Let $H = \{z = x + iy : \operatorname{Re} z > 0\}$ be the right half-plane. It is shown [9, Example 7.2, p. 29] that

$$\lambda_H(z)|dz| = \frac{|dz|}{x}.$$

The following result and its corollary relates to functions mapping the disk respectively into H or a convex domain. By incorporating the hyperbolic metric, they improve the results of Aizenberg [7] in the sense that the reciprocal of the hyperbolic metric for a convex domain (cf. Theorem 1.4) yields a sharper estimate than the Euclidean distance.

Theorem 2.1 *Let $g(z) = \sum_{n=0}^{\infty} a_n z^n \in H$ for all $z \in U$. Then*

$$\sum_{n=1}^{\infty} |a_n z^n| \le \frac{1}{\lambda_H(a_0)} = d(a_0, \partial H)$$

when $|z| \le \rho = 1/3$. This bound is sharp.

Proof. Let $F(z) = (\operatorname{Re} a_0)((1 + z)/(1 - z))$. Then $h(z) = g(z) - i \operatorname{Im} a_0 = \sum_{n=1}^{\infty} a_n z^n + \operatorname{Re} a_0$ is subordinate to F. Since F is conformal, it follows that

$$d_H(h(z), \operatorname{Re} a_0) \le d_H(F(z), \operatorname{Re} a_0) = d_U(z, 0) = \log \frac{1 + |z|}{1 - |z|}.$$

Let $w(z) = \sum_{n=1}^{\infty} |a_n| z^n + \operatorname{Re} a_0$. Then by Herglotz formula [16], we deduce that $w(|z|) \le F(|z|)$. Hence

$$d_H(w(|z|), \operatorname{Re} a_0) = \log \frac{w(|z|)}{\operatorname{Re} a_0} \le d_H(F(|z|), \operatorname{Re} a_0) = \log \frac{1 + |z|}{1 - |z|}.$$

Consequently, $w(|z|) \le (\mathrm{Re}\, a_0)((1+|z|)/(1-|z|))$.

If $|z| \le 1/3$, then $w(|z|) \le 2\mathrm{Re}\, a_0$, and thus

$$w(|z|) - \mathrm{Re}\, a_0 \le \mathrm{Re}\, a_0 = \frac{1}{\lambda_H(a_0)} = d(a_0, \partial H). \qquad \square$$

Corollary 2.2 *Let $g(z) = \sum_{n=0}^{\infty} a_n z^n \in \Omega$, where Ω is a convex domain. Then*

$$\sum_{n=1}^{\infty} |a_n z^n| \le \frac{1}{\lambda_\Omega(a_0)} \le d(a_0, \partial\Omega),$$

for $|z| \le 1/3$. This result is sharp.

P r o o f. Let $\zeta \in \partial\Omega$ be nearest to a_0. Further let T_ζ be the tangent line at ζ, and H_ζ the half-plane containing Ω. Then $g(z) \in H_\zeta$. Choose t real so that $(H_\zeta - \zeta)\, e^{it} = H$, the right half-plane. Let $\Omega_1 = (\Omega - \zeta)\, e^{it}$. Hence $(g(z) - \zeta)\, e^{it} \in \Omega_1 \subset H$ and $(a_0 - \zeta)\, e^{it} = |\, a_0 - \zeta|$.

Let $f(z) = \sum_{n=0}^{\infty} b_n z^n$ be the conformal map from U onto Ω_1, with $f(0) = |a_0 - \zeta|$ and $b_1 > 0$. By Lemma 4 in [1], $|a_n| \le |b_1|$ for all $n > 1$. Consequently for $|z| \le 1/3$,

$$\sum_{n=1}^{\infty} |a_n z^n| \le \frac{b_1}{2} = \frac{1}{\lambda_{\Omega_1}(f(0))} = \frac{1}{\lambda_\Omega(a_0)}.$$

As $f(z)$ is subordinate to $f(0)((1+z)/(1-z))$, Herglotz formula yields $|b_1| \le 2f(0)$. Hence

$$\begin{aligned}\sum_{n=1}^{\infty} |a_n z^n| &\le \frac{1}{\lambda_\Omega(a_0)} \le f(0) = |a_0 - \zeta| \\ &= \frac{1}{\lambda_H\left((a_0 - \zeta)\, e^{it}\right)} = d\left((a_0 - \zeta)\, e^{it}, \partial H\right) = d(a_0, \partial\Omega). \qquad \square\end{aligned}$$

2.2 Slit map

Let $P = \{z : |\arg z| < \pi\}$. From (1.5), it is shown in [9, Example 7.7, p. 31] that

$$\lambda_P(z)|dz| = \frac{|dz|}{2|\sqrt{z}|\mathrm{Re}\, \sqrt{z}} = \frac{|dz|}{2|z| \cos\theta/2} \ge \frac{|dz|}{2|z|}, \quad \left(z = re^{i\theta}\right).$$

This theorem is about functions mapping the disk into P.

Theorem 2.3 *If $h(z) = \sum_{n=0}^{\infty} a_n z^n \in P$ for all $z \in U$, then*

$$\sum_{n=1}^{\infty} |a_n|\, |z|^n \le \frac{1}{2\lambda_P(|a_0|)} = d(|a_0|, \partial P)$$

for $|z| \le \rho = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 0.171\,57$. This result is sharp.

P r o o f. Let

$$F(z) = |a_0| \left(\frac{1+z}{1-z}\right)^2.$$

Then $F(z) \in P$ and

$$d_P(F(z), F(0)) = d_U(z, 0) = \log \frac{1+|z|}{1-|z|}.$$

Now

$$d_P(F(|z|), F(0)) = \int_{F(0)}^{F(|z|)} \lambda_P(z)|dz| = \int_{F(0)}^{F(|z|)} \frac{1}{2|z|}|dz| = \frac{1}{2}\log\frac{F(|z|)}{F(0)}.$$

Thus

$$\frac{F(|z|)}{F(0)} = \left(\frac{1+|z|}{1-|z|}\right)^2.$$

When $\rho \le \frac{\sqrt{2}-1}{\sqrt{2}+1} = 0.171\,57$, $F(|z|) - F(0) \le F(0) = \frac{1}{2\lambda_P(F(0))} = d(F(0), \partial P)$.

De Brange's Theorem [14] implies that $\sum_{n=0}^{\infty} |a_n z^n| \le F(|z|)$, which yields the desired result. □

Corollary 2.4 *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ map U conformally into a simply connected domain Ω. Then*

$$\sum_{n=1}^{\infty} |a_n|\,|z|^n \le \frac{1}{2\lambda_\Omega(|a_0|)} \le d(|a_0|, \partial\Omega) \le \frac{2}{\lambda_\Omega(|a_0|)|}$$

when $|z| \le \rho = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 0.171\,57$.

P r o o f. Let $\zeta \in \partial\Omega$ be closest to a_0 and let $e^{it}(\Omega - \zeta) = \Omega_1$, where t is chosen so that $e^{it}(a_0 - \zeta) > 0$ and the origin on $\partial\Omega_1$ is closest to $b_0 = e^{it}(a_0 - \zeta)$. Let

$$g(z) = e^{it}(h(z) - \zeta) = \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} e^{it}(a_n - \zeta)z^n$$

and

$$f(z) = b_0\left(\frac{1+z}{1-z}\right)^2 = \sum_{n=0}^{\infty} B_n z^n = b_0\left(1 + 4\sum_{n=1}^{\infty} n z^n\right).$$

From de Brange's Theorem [14], it follows that

$$|b_n| \le n\,|b_1| = \frac{|b_1|}{4\,|b_0|}(4\,|b_0|\,n).$$

Thus Theorem 2.3, Theorem 1.4, along with the fact that $\lambda_{\Omega_1}(|b_0|) = \lambda_\Omega(|a_0|)$ [9, p. 36], yield

$$\begin{aligned}\sum_{n=1}^{\infty} |b_n z^n| &\le \frac{|b_1|}{4\,|b_0|}\sum_{n=1}^{\infty} |B_n z^n| \le \frac{|b_1|}{4\,|b_0|}\frac{1}{2\lambda_P(|b_0|)} \\ &= \frac{|b_1|}{4} = \frac{1}{2\lambda_{\Omega_1}(|b_0|)} = \frac{1}{2\lambda_\Omega(|a_0|)} \le d(|a_0|, \partial\Omega).\end{aligned}$$

□

2.3 Exterior of the unit disk

Let $U^* = \{z : |z| > 1\}$. From (1.5), it is easily deduced that

$$\lambda_{U^*}(z)|dz| = \frac{|dz|}{|z|\log|z|}.$$

Theorem 2.5 *Let $a_0 > 1$, and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in U^*$ for all $z \in U$. If $|z| \le 1/3$, then*

(a) $\log\left(\frac{\sum_{n=0}^{\infty} |a_n z^n|}{a_0}\right) \le \frac{1}{\lambda_H(\log a_0)} = d(\log a_0, \partial H)$,

(b) $\sum_{n=1}^{\infty} |a_n z^n| \le \frac{2}{\lambda_{U^*}(a_0)}$, *provided* $a_0 \le 2$.

Proof. Let $F(z) = \exp\left[\frac{(\log a_0)(1+z)}{(1-z)}\right]$ be a universal covering of U^*. Note that

$$g(z) = \sum_{n=0}^{\infty} |a_n|\, z^n$$

is also in U^*. Then

$$\begin{aligned} d_{U^*}(g(|z|), a_0) &= \log\left[\frac{\log g(|z|)}{\log a_0}\right] \le d_{U^*}(F(|z|), a_0) \\ &= \log\left[\frac{\log F(|z|)}{\log a_0}\right] = \log\frac{1+|z|}{1-|z|}. \end{aligned}$$

Thus

$$\log g(|z|) \le \log F(|z|) = \log a_0^{\frac{1+|z|}{1-|z|}}.$$

When $|z| \le 1/3$,

$$\begin{aligned} \log g(|z|) - \log a_0 &\le \log F(|z|) - \log a_0 \le \log a_0 \\ &= \frac{1}{\lambda_H(\log a_0)} = d(\log a_0, \partial H) = \frac{1}{|a_0|\lambda_{U^*}(a_0)}. \end{aligned}$$

In addition,

$$F(|z|) - a_0 \le a_0(a_0 - 1)$$

whenever $|z| \le 1/3$. Since $g(|z|) \le F(|z|)$, the condition $a_0 \le 2$ yields

$$\sum_{n=1}^{\infty} |a_n z^n| \le \frac{2}{\lambda_{U^*}(a_0)}. \qquad \square$$

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